Incorporating Losses into the ED Problem

\[
\text{minimize} \quad \sum_{i=1}^{m} (\alpha_i + \beta_i P_i^G + \gamma_i (P_i^G)^2)
\]

subject to

\[
\sum_{i=1}^{m} P_i^G = P^D + \text{losses}
\]

\[
P_i^G \leq P_i \leq \bar{P}_i^G
\]

The system losses appear because of the terms \(V_i V_k G_{ik} \cos(\Theta_i - \Theta_k)\) in the power flow equations. Actually, losses

\[
\text{losses} = \sum_{i,j \leq k} V_i V_k G_{ik} \cos(\Theta_i - \Theta_k)
\]

\[
= \sum_{i=1}^{n} \left( \sum_{k=1}^{n} V_i V_k G_{ik} \cos(\Theta_i - \Theta_k) \right)
\]

The problem is that we don't know \(V_i, V_k, \Theta_i, \Theta_k\) a priori unless we solve the power flow equations for given \(P_1^G, P_2^G, \ldots, P_m^G, V_2, \ldots, V_m, P^D, \Theta^D, \ldots, \Theta^D)\).
However, it is clear that since the unknown $v_i$'s and $\theta_i$'s are a function of the problem data, then the losses are clearly a function of $P_2, \ldots, P_m$

\[
\text{losses} = P^L (P_2^G, \ldots, P_m^G)
\]

In reality we could also write them to be a function of $P_1^G$ but as we argued before, $P_1^G$ cannot be specified independently of $P_2^G, \ldots, P_m^G$; i.e., $P_1^G = f(P_2^G, \ldots, P_m^G)$.

Hence the dependence of $P^L$ on $P_2^G, \ldots, P_m^G$ only.

If we disregard generation limits, we then have

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} (\alpha_i + \beta_i P_i^G + \gamma_i (P_i^G)^2) \\
\text{subject to} & \quad \sum_{i=1}^{m} P_i^G = P^D + P^L (P_2^G, \ldots, P_m^G)
\end{align*}
\]
In the problem above, it should be clear that $P^L(\cdot)$ is in general not available. (it would involve finding a closed-form solution for the power flow equations!)

Next, we explore how losses affect the solution to the ED problem. (No constraints on generator power output)

Introduce a Lagrangian $\lambda$ and write

$$\text{minimize } \sum_{i=1}^{m} (\alpha_i + \beta_i P_i^G + \gamma_i (P_i^G)^2) + \lambda (P^D - \sum_{i=1}^{m} P_i^G - P^L(\cdot))$$

$$\tilde{C}(P_1^G, ..., P_m^G, \lambda)$$

$$\nabla \tilde{C}(P_1^G, P_2^G, ..., P_m^G, \lambda) = 0$$

$\Rightarrow$ (Next page)
\[ \beta_1 P_1^G + 2 \gamma_1 P_1^G - \lambda = 0 \]
\[ \beta_2 P_2^G + 2 \gamma_2 P_2^G - \lambda \left(1 - \frac{\partial P^L}{\partial P_2^G} (P_2^G, \ldots, P_m^G)\right) = 0 \]
\[ \vdots \]
\[ \beta_m P_m^G + 2 \gamma_m P_m^G - \lambda \left(1 - \frac{\partial P^L}{\partial P_m^G} (P_2^G, \ldots, P_m^G)\right) = 0 \]
\[ \sum_{i=1}^m P_i^G = P^D + P^L (P_2^G, \ldots, P_m^G) = 0 \]

Define:
\[ L_i (P_2^G, \ldots, P_m^G) = \begin{cases} 1, & \text{if } i = 1 \\ \frac{1}{1 - \frac{\partial P^L}{\partial P_i^G} (P_2^G, \ldots, P_m^G)}, & \text{if } i \neq 1 \end{cases} \]

These are referred to as penalty factors.

Penalty factors vary with the solution (tie operating point), but they are relatively constant, so if we assume them to be constant, we have that the losses are linear with \( P_2, \ldots, P_m \) and we can write:
\[ L_i (\beta_i + 2 \gamma_i P_i^G) = \lambda, \quad i = 2, \ldots, m \]
\[ \sum_{i=1}^m P_i^G = P^D + \sum_{i=1}^m \left(1 - \frac{1}{L_i}\right) P_i^G \]
Rearranging, we obtain
\[ L_i \cdot (\beta_i + 2\delta_i P_i^g) = \lambda, \quad i = 2, \ldots, m \]
\[ \sum_{i=1}^{m} \frac{1}{L_i} P_i^g = P^D \]

and we could solve for \( P_i \) and \( \lambda \) to obtain the solution.

Define \( \tilde{P}_i^G = \frac{1}{L_i} P_i^G \), \( \tilde{\beta}_i = L_i \beta_i \), \( \tilde{\delta}_i = L_i \delta_i \).

Then, we can rewrite the equations above as:
\[ \tilde{\beta}_i + 2\tilde{\delta}_i \tilde{P}_i^G = \lambda, \]
\[ \sum_{i=1}^{m} \tilde{P}_i^G = P^D \]

which is of the same form as the equations we would get in the lossless case.

We can now see what's the effect of the losses in the solution.
Take two generators $i, j$ with identical cost functions, i.e., $\alpha_i = \alpha_j$, $\beta_i = \beta_j$, $\delta_i = \delta_j$

In the lossless case, we have

\[
\begin{align*}
\beta_i + 2\delta_i \tilde{P}_i^6 &= \lambda \\
\beta_j + 2\delta_j \tilde{P}_j^6 &= \lambda
\end{align*}
\]

\[\Rightarrow \begin{cases} 
\tilde{P}_i^6 = \tilde{P}_j^6
\end{cases}
\]

Clearly both produce the same amount of power.

In the lossy case:

\[
\begin{align*}
L_i (\beta_i + 2\delta_i \tilde{P}_i^6) &= \lambda \\
L_j (\beta_j + 2\delta_j \tilde{P}_j^6) &= \lambda
\end{align*}
\]

\[\Rightarrow \begin{cases} 
L_i (\beta_i + 2\delta_i \tilde{P}_i^6) = L_j (\beta_j + 2\delta_j \tilde{P}_j^6)
\end{cases}
\]

or using the tilde variables:

\[
\begin{align*}
\tilde{\beta}_i + 2\tilde{\delta}_i \tilde{\tilde{P}}_i^6 &= \tilde{\beta}_j + 2\tilde{\delta}_j \tilde{\tilde{P}}_j^6 \\
\Rightarrow \begin{cases} 
\tilde{P}_i^6 = \frac{\tilde{\beta}_j - \tilde{\beta}_i}{2\tilde{\delta}_i} + \frac{\tilde{\delta}_j}{\tilde{\delta}_i} \tilde{\tilde{P}}_j^6
\end{cases}
\end{align*}
\]

Thus if $L_i \neq L_j \Rightarrow \tilde{P}_i^6 \neq \tilde{P}_j^6$

Moreover:

\[
\tilde{P}_i^6 < \tilde{P}_j^6 \text{ if } L_i > L_j
\]
Define \( p_i = p_j = B \) and \( \gamma_i = \gamma_j = \gamma_j \).

\[
\frac{P_i^G}{L_i} = \frac{B(L_j - L_i)}{2\gamma L_i^2} + \frac{L_j^2}{L_i^2} \cdot \frac{P_j^G}{L_j}
\]

\[
P_i^G = \frac{B(L_j - L_i)}{2\gamma L_i^2} + \frac{L_j}{L_i} \cdot P_j^G < P_j^G
\]

Since \( L_j - L_i < 0 \) and \( \frac{L_j}{L_i} < 1 \).